Matrix Lie Groups

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1 Introduction

This mini-course will focus on understanding matrix Lie groups. Lie groups are groups with a so-called "smooth structure". This means that locally we can smoothly parametrize our group using a finite number of parameters i.e. for a Lie group G, each element g has an open neighbourhood U such that $U \cong \mathbb{R}^n$. Now, one consequence of this smooth structure, is that we can understand things infinitesimally. We will make sense of what it means to differentiate the group structure. This leads to the notion of a Lie algebra. In the first part, we will introduce the above notions properly. In the second half, we will try to answer the following question.

Question 1.1. Given a Lie algebra i.e. the infinitesimal structure of a Lie group, can we recover the original group?

In order to properly define what a smooth structure is, one would need an entirely different mini-course. However, most of the theory can be understood without understanding the abstract definition of a smooth structure if one instead focuses on matrix Lie groups. These are Lie groups which can be realised as closed subgroups of $\mathrm{GL}(n, \mathbb{R})$ for some $n \in \mathbb{R}$.

The contents of these notes are based on chapters 1, 2, 3 and 5 of [Hal15]. Whenever relevant, additional or alternative references are provided.

2 Part 1

2.1 Matrix Lie Groups

Recall that the general linear group $\mathrm{GL}(n, \mathbb{C})$ i.e. the group of invertible $n \times n$ matrices with coefficients in $\mathbb{C}$ may be defined as

$$\mathrm{GL}(n, \mathbb{C}) = \{A \in M_n(\mathbb{C}) \mid \det(A) \neq 0\}.$$

Note that $M_n(\mathbb{C})$ is a topological space, since we can canonically identify it with $\mathbb{C}^{n^2}$. Thus, $\mathrm{GL}(n, \mathbb{C})$ as an open subset of $\mathbb{C}^{n^2}$ is topological space with the subset topology.

Definition 2.1. A *matrix Lie group* is a closed subgroup G of $\mathrm{GL}(n, \mathbb{C})$ i.e. G is a subgroup of $\mathrm{GL}(n, \mathbb{C})$ and closed as a subset of $\mathrm{GL}(n, \mathbb{C})$. ♢

Example 2.2 (General linear group)*.* The group $\mathrm{GL}(n, \mathbb{C})$ itself is closed in $\mathrm{GL}(n, \mathbb{C})$. So, it is a matrix Lie group. △

Example 2.3 (General linear group with coefficients in $\mathbb{R}$)*.* The group $\mathrm{GL}(n, \mathbb{R})$ is a matrix Lie group. To see this, note that $\mathrm{GL}(n, \mathbb{R}) = \mathrm{GL}(n, \mathbb{C}) \cap \mathfrak{gl}(n, \mathbb{R})$, where $\mathfrak{gl}(n, \mathbb{R})$ is the real vector subspace of $\mathfrak{gl}(n, \mathbb{C})$ given by $a_i \mapsto a_i + 0i$. △

Example 2.4*.* Let G be matrix Lie group inside $\mathrm{GL}(n, \mathbb{C})$. Then, every closed subgroup H of G is a matrix Lie group. Note that as G is closed in $\mathrm{GL}(n, \mathbb{C})$, H is also closed as a subset of $\mathrm{GL}(n, \mathbb{C})$. △

Example 2.5 (Orthogonal group)*.* Let $n \in \mathbb{N}$. The orthogonal group $O(n)$ is given by the matrices in $\mathrm{GL}(n, \mathbb{C})$ that preserve the standard inner product.

$$\mathrm{O}(n) = \{A \in \mathrm{GL}(n, \mathbb{R}) \mid A^T A = I\}$$

Note this is closed, since the map $A \mapsto A^T A$ is continuous. △

Example 2.6 (Special linear group)*.* The special linear group is given by the matrices with determinant 1

$$\mathrm{SL}(n) = \{A \in \mathrm{GL}(n, \mathbb{C}) \mid \det(A) = 1\}.$$

△

Example 2.7 (Intersections of matrix Lie groups)*.* Let G, H be two matrix Lie groups contained in $\mathrm{GL}(n, \mathbb{C})$. Then, their intersection $G \cap H$ is a matrix Lie group. △

Example 2.8 (Special orthogonal group)*.* The special orthogonal group is given by intersecting the orthogonal and the special linear group.

$$\mathrm{SO}(n) = \{A \in \mathrm{GL}(n, \mathbb{R}) \mid \det(A) = 1, A^T A = I\}.$$

Another way to think of this group is as the rotations of n-dimensional space. ($A^T A = I$ says that these transformations preserve length with respect to the standard inner product and $\det(A) = 1$ excludes reflections). △

Example 2.9*.* The unitary group is given by:

$$\mathrm{U}(n) = \{A \in \mathrm{GL}(n, \mathbb{C}) \mid A^\dagger A = I\}$$

and the special unitary group by

$$\mathrm{SU}(n) = \{A \in \mathrm{GL}(n, \mathbb{C}) \mid A^\dagger A = I, \det(A) = 1\}$$

△

2.2 The matrix exponential

Definition 2.10. Let $X \in \mathfrak{gl}(n, \mathbb{C})$. Then, the matrix exponential is defined by

$$\exp(X) := \sum_{k=0}^{\infty} \frac{X^k}{k!}$$

♢

In exercise 10, you will show that this map is well-defined and continuous.

Proposition 2.11. *The matrix exponential satisfies the following properties for all $X, Y \in \mathfrak{gl}(n, \mathbb{C})$ and $C \in \mathrm{GL}(n, \mathbb{C})$:*

(i) $\exp(CXC^{-1}) = C\exp(X)C^{-1}$*;*
(ii) $\exp(X)$ *is invertible with inverse* $\exp(-X)$*;*
(iii) $\det(\exp(X)) = \exp(\mathrm{tr}(X))$*;*
(iv) if $XY = YX$*, then* $\exp(X+Y) = \exp(X)\exp(Y)$*.*

Proof. See exercise 4. □

Recall that $\log(x) = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{(x-1)^k}{k}$ converges absolutely for $\|x\| < 1$. As you can show in exercise 10, this means that

$$\log(A) := \sum_{k=1}^{\infty}(-1)^{k+1}\frac{(A-I)^k}{k}$$

converges for $\|A - I\| < 1$ and gives a well-defined continuous function.

Theorem 2.12. *In case,* $\|A - I\| < 1$*, we have*

$$\exp(\log(A)) = A$$

and in case, $\|X\| < \log 2$*, we have*

$$\log(e^X) = X$$

Proof. Exercise 4. □

Proposition 2.13. *Let* $X, Y \in \mathfrak{gl}(n, \mathbb{C})$*. Then, we have*

$$(\exp(X/m)\exp(Y/m))^m \to \exp(X+Y)$$

as $m \to \infty$*.*

Proof. Exercise. □

2.3 Lie algebras

Definition 2.14. An abstract *Lie algebra* over $\mathbb{R}$ (or $\mathbb{C}$) consists of a vector space $\mathfrak{g}$ over $\mathbb{R}$ (or $\mathbb{C}$) together with a bilinear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that

(i) $[\cdot,\cdot]$ is anti-symmetric i.e. for all $X, Y \in \mathfrak{g}$, we have

$$[X, Y] = -[Y, X]$$

(ii) $[\cdot,\cdot]$ satisfies the Jacobi identity i.e. for all $X, Y, Z \in \mathfrak{g}$, we have

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

◊

Example 2.15. Let A be an associative algebra. Define a bilinear operation

$$[a,b] = ab - ba$$

with $a, b \in A$. It is a straightforward computation to check that A together with this operation is a Lie algebra. Let us write $\mathfrak{gl}(n,\mathbb{C})$ for the algebra of $n \times n$ matrices with coefficients in $\mathbb{C}$. With the described bracket, $\mathfrak{gl}(n,\mathbb{C})$ becomes a Lie algebra. Similarly, $\mathfrak{gl}(n,\mathbb{R})$ is a Lie algebra. △

Definition 2.16. Let $\mathfrak{g}$ be a Lie algebra. A Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is a vector subspace of $\mathfrak{g}$ closed the bracket of $\mathfrak{g}$ i.e. $[\mathfrak{h},\mathfrak{h}] \subseteq \mathfrak{h}$. ♢

Proposition 2.17. *A Lie subalgebra is a Lie algebra.*

Example 2.18. The subspace $\mathfrak{sl}(n,\mathbb{C}) = \{X \in \mathfrak{gl}(n,\mathbb{C}) \mid \mathrm{tr}(X) = 0\}$ is a Lie subalgebra of $\mathfrak{gl}(n,\mathbb{C})$, since

$$\mathrm{tr}([X,Y]) = \mathrm{tr}(XY) - \mathrm{tr}(YX) = 0,$$

by the cyclic property of the trace. △

Example 2.19. We will now enumerate a few typical examples of Lie algebras that all arise as subalgebra of $\mathfrak{gl}(n,\mathbb{C})$ and $\mathfrak{gl}(n,\mathbb{R})$. Verifying that these are indeed Lie subalgebras is an exercise. For a complex matrix X, we write X^* its hermitian conjugate.

1. $\mathfrak{u}(n) = \{X \in \mathfrak{gl}(n,\mathbb{C}) \mid X^* = -X\}$.
2. $\mathfrak{o}(n) = \{X \in \mathfrak{gl}(n,\mathbb{R}) \mid X^T = -X\}$.
3. $\mathfrak{su}(n) = \mathfrak{sl}(n,\mathbb{C}) \cap \mathfrak{u}(n)$.
4. $\mathfrak{so}(n) = \mathfrak{sl}(n,\mathbb{R}) \cap \mathfrak{o}(n)$

△

Definition 2.20. Let $\mathfrak{g}$ be a Lie algebra. Then, for $X \in \mathfrak{g}$, we define

$$\begin{aligned} \mathrm{ad}_X : \mathfrak{g} &\to \mathfrak{g} \\ Y &\mapsto [X,Y]. \end{aligned}$$

This defines the adjoint map $\mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ given by $X \mapsto \mathrm{ad}_X$. ♢

Remark 2.21. In terms of the adjoint map, the Jacob identity is given by

$$\mathrm{ad}_X[Y,Z] = [\mathrm{ad}_X(Y),Z] + [Y,\mathrm{ad}_X(Z)].$$

In this form, the Jacobi reflects that ad_X satisfies a type of product rule when applied to the binary operation $[\,\cdot\,,\,\cdot\,]$. Furthermore, another way of writing the Jacobi identity is as

$$\mathrm{ad}_{[X,Y]}(Z) = [\mathrm{ad}_X, \mathrm{ad}_Y](Z).$$

In this form, the Jacobi identity reflects the fact that ad is a Lie algebra homomorphism. ♡

2.4 The Lie algebra of a matrix Lie group

Definition 2.22. Let $G \subseteq \mathrm{GL}(n, \mathbb{C})$ be a matrix Lie group. The Lie algebra $\mathfrak{g}$ of G is defined as those $X \in \mathfrak{gl}(n, \mathbb{C})$ such that $e^{tX} \in G$ for all $t \in \mathbb{R}$. $\diamondsuit$

Intuitively, the Lie algebra captures all infinitesimal directions at the identity that are tangent to G.

Proposition 2.23. *Let G be a matrix Lie group with Lie algebra $\mathfrak{g}$. Let $X, Y \in \mathfrak{g}$, then the following holds.*

(i) For all $A \in G$, $AXA^{-1} \in \mathfrak{g}$.
(ii) $sX \in \mathfrak{g}$ for all $s \in \mathbb{R}$.
(iii) $X + Y \in \mathfrak{g}$.
(iv) $XY - YX \in \mathfrak{g}$.

Proof. Point one follow from

$$\exp(tAXA^{-1}) = A\exp(tX)A^{-1}.$$

The second point follows from

$$\exp(t(sX)) = \exp((ts)X).$$

The third point follows from the fact that

$$\exp(t(X+Y)) = \lim_{m\to\infty}(\exp(tX/m)\exp(tY/m))^m$$

and that G is closed. Let us now proof the final point. Firstly, remark that point 2 and 3 imply that $\mathfrak{g}$ is a vector subspace of $\mathfrak{gl}(n, \mathbb{C})$. In particular, it is a closed subspace as a topological space.

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\exp(tX)Y\exp(-tX)) = XY - YX.$$

By point 1, $\exp(tY)X\exp(-tY) \in \mathfrak{g}$. So, as $\mathfrak{g}$ is closed, we see that the limit

$$\lim_{t\to 0}\frac{\exp(tX)Y\exp(-tX) - Y}{t} = XY - YX$$

lies in $\mathfrak{g}$. $\square$

Theorem 2.24. *The Lie algebra $\mathfrak{g}$ associated to a matrix Lie group G with the Lie bracket*

$$[X, Y] = XY - YX$$

is indeed a Lie algebra.

As the notation already suggested, the examples of Lie groups and Lie algebras that we have introduced so far are indeed associated to each other.

Proposition 2.25. *The Lie algebra of $\mathrm{GL}(n, \mathbb{C})$ is $\mathfrak{gl}(n, \mathbb{C})$ and the Lie algebra of $\mathrm{SL}(n, \mathbb{C})$ is $\mathfrak{sl}(n, \mathbb{C})$.*

Proof. The first of these two is clear. For $\mathrm{SL}(n, \mathbb{C})$, we need that

$$1 = \det(\exp(tX)) = \exp(t \operatorname{tr}(X)).$$

Then, upon differentiating both sides by t, we find that $\operatorname{tr}(X) = 0$. Clearly, this condition is also sufficient. Thus, the Lie algebra of $\mathrm{SL}(n, \mathbb{C})$ is $\mathfrak{sl}(n, \mathbb{C})$. $\square$

Definition 2.26. The exponential map of a Lie group G is the map

$$\exp : \mathfrak{g} \to G.$$

$\diamondsuit$

In general, this map is neither surjective nor injective (See exercises 3, 5 and 15). However, it does fully determine the local structure of the Lie group.

Theorem 2.27. *There exists some* $0 < \epsilon < \log(2)$ *such that* $\exp$ *restricted to* $U_\epsilon := B_\epsilon(0) \subseteq \mathfrak{g}$ *is a local homeomorphism* $U_\epsilon \to \exp(U_\epsilon) =: V_\epsilon$.

Proof. Let $D = \mathfrak{g}^\perp$ with respect to the standard metric. Then, define

$$\Phi : \mathfrak{g} \oplus D \to \mathrm{GL}(n, \mathbb{C})$$

by

$$(X, Y) \mapsto \exp(X) \exp(Y).$$

Note that at the origin, the differential of Φ is given by

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Phi(tX, 0) = X$$

and

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Phi(0, tY) = Y.$$

So, the derivative Φ at 0 is the identity. Then, by the inverse function theorem, we have neighborhood of the origin such that Φ is invertible. Let us call the local inverse Ψ. In particular, we can find an $\epsilon > 0$ such that U_ϵ lies inside this neighborhood. Then, to show that $\exp = \Phi|_{U_\epsilon \oplus 0}$ is a homeomorphism, it suffices to show that its image is open in G. Suppose this is not the case. Then, there exists a sequence A_m of matrices in G such that $A_m \to I$, but $\log A_m \notin \mathfrak{g}$. For m large enough, we have A_m in the domain of Ψ. Thus,

$$A_m = \exp(X_m) \exp(Y_m).$$

Furthermore, since A_m and $\exp(X_m)$ lie in G, it follows that $\exp(Y_m) \in G$. Now, remark that $Y_m / \|Y_m\|$ defines a sequence on the unit sphere of D. Since the unit sphere is compact, we may choose a subsequence of Y_m such that $Y_m / \|Y_m\|$ converges to Y inside the unit sphere of D. Note since $A_m \to I$ and Φ is a local homeomorphism, $Y_m \to 0$. In addition, there exist integers k_m such that $k_m \|Y_m\| \to t$. (Take k_m to be the rounding of $t / \|Y_m\|$. Then, $|k_m \|Y_m\| - t| \leq \|Y_m\| \to 0$.) So, as G is closed, we find that

$$\exp(tY) = \lim_{m \to \infty} \exp(k_m \|Y_m\| (Y_m / \|Y_m\|)) = \lim_{m \to \infty} \exp(Y_m)^{k_m}$$

lies in G. However, then $Y \neq 0$ would be in $\mathfrak{g}$. This is a contradiction. $\square$

Proposition 2.28. *For a connected matrix Lie group G, every element A may be written as*

$$A = \exp(X_1) \dots \exp(X_n)$$

for some $X_1, \dots, X_n \in \mathfrak{g}$.

Proof. Since G is connected, we can find a path $I : [0,1] \to G$ from I to A. Now, from the previous proof, we have an open neighborhood V_ϵ such that $\exp : U_\epsilon \to V_\epsilon$ is a homeomorphism. Furthermore, $I(t)V_\epsilon$ is an open neighborhood of $I(t)$. In particular, $I^{-1}(I(t)V_\epsilon)$ defines an open cover of $[0,1]$. Since $[0,1]$ is compact, we can find a finite subcover T_λ of $[0,1]$. We can now find a finite number points $0 = t_0 < t_1 < \dots < t_n = 1$ such that $[t_i, t_{i+1}] \subseteq T_\lambda$. Then, we see that

$$I(t_i) = I(\lambda) \exp(X_i)$$

for some $X_i \in \mathfrak{g}$ and

$$I(t_{i+1}) = I(\lambda) \exp(X'_{i+1}),$$

for some $\exp(X'_{i+1}) \in \mathfrak{g}$. So, we see that $I(t_i)^{-1} I(t_{i+1}) = \exp(-X_i) \exp(X'_{i+1})$. Now, we may write

$$\begin{aligned} A = I(1) &= I(0) I(t_1)^{-1} I(t_1) \dots I(t_{n-1}) I(t_{n-1})^{-1} I(t_n) \\ &= \exp(X'_0) \exp(-X_1) \exp(X'_1) \dots \exp(-X_{n-1}) \exp(X_n). \end{aligned}$$

□

We say that a Lie algebra $\mathfrak{g}$ is commutative if $[X, Y] = 0$ for all $X, Y \in \mathfrak{g}$. The commutativity of the Lie group is in fact reflected in its Lie algebra.

Proposition 2.29. *Let G be a matrix Lie group with Lie algebra $\mathfrak{g}$. Suppose G is commutative, then $\mathfrak{g}$ is commutative.*

On the other hand, if G is connected, then converse also holds.

Proof. Remark that

$$\begin{aligned} [X, Y] &= \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp(sX) \exp(tY) \exp(-sX) \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp(tY) = 0. \end{aligned}$$

Now, for the converse, note that if $[X, Y] = 0$, then

$$\exp(X) \exp(Y) = \exp(X + Y) = \exp(Y + X) = \exp(X) \exp(Y).$$

Now, in case G is connected, any element of G may be written as a product of exponentials. So, we see that in this case G is commutative. □

2.5 Exercises

Exercise 1

Consider the matrices

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad C = A + B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(a) Compute $\exp(A)$, $\exp(B)$ and $\exp(C)$.

(b) Conclude that $\exp(A + B) \neq \exp(A)\exp(B)$.

Exercise 2

(a) Do the straightforward computation suggested in example 2.15.

(b) Show that the vector spaces in example 2.19 are indeed Lie algebras.

Exercise 3

Let $\theta \in \mathbb{R}$. Consider the matrix

$$A_\theta = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}$$

(a) Compute $\exp(A_\theta)$.

(b) Conclude $\mathrm{Lie}(\mathrm{SO}(2)) = \{A_\theta : \theta \in \mathbb{R}\}$.

Exercise 4

A common trick to prove identities involving matrices is to first show that an equation holds on a dense subset of a space and to then extend the equation using continuity. This exercises illustrates this concept.

(a) Show that $n \times n$ matrices with n distinct eigenvalues are dense in $M_n(\mathbb{C})$.

Hint: Every matrix is similar to an upper triangular one.

(b) Conclude that diagonalizable matrices are dense in $n \times n$ matrices.

Let $X, Y \in M_n(\mathbb{C})$ and let $C \in \mathrm{GL}(n, \mathbb{C})$.

(c) Show that

$$\exp(CXC^{-1}) = C\exp(X)C^{-1}.$$

(d) Prove that $\exp(X)$ is invertible with inverse $\exp(-X)$.

(e) Prove that $\det(\exp(X)) = \exp(\mathrm{tr}(X))$.

(f) Prove that if $XY = YX$, then $\exp(X + Y) = \exp(X)\exp(Y)$.

Hint: Show that two commuting matrices with n distinct eigenvalues can be diagionalised with respect to the same basis.

(g) Prove theorem 2.12.

Exercise 5

Consider the group

$$\mathrm{Aff}(1,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}(2,\mathbb{R}) \,\middle|\, a, b \in \mathbb{R} \right\}.$$

(a) Show that
$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} ax+b \\ 1 \end{pmatrix}.$$
Explain the name of the group.

(b) Compute
$$\exp \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{X}, \quad \exp \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{Y}.$$
Conclude that $\mathfrak{aff}(1,\mathbb{R}) = \mathrm{Lie}(\mathrm{Aff}(1,\mathbb{R})) = \langle X, Y \rangle$.

(c) Show that $[X, Y] = Y$.

Exercise 6

(a) Let $\mathfrak{g}$ be a two-dimensional non-commutative Lie algebra. Show that there exists a basis $\{x, y\}$ for $\mathfrak{g}$ such that $[x, y] = x$.

(b) Answer question 1.1 in the case of two-dimensional Lie algebras.

Exercise 7

The Heisenberg group is defined as

$$\mathrm{Heis} = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}(3,\mathbb{R}) \,\middle|\, a, b, c \in \mathbb{R} \right\}$$

(a) Check that Heis is closed under matrix multiplication and inversion.

(b) Check that $\mathrm{Heis} \subset \mathrm{GL}(3,\mathbb{R})$ is closed.

(c) Show that
$$\mathfrak{heis} = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{gl}(3,\mathbb{R}) \,\middle|\, a, b, c \in \mathbb{R} \right\}$$
is the Lie algebra of $\mathfrak{heis}$.

(d) Show that the exponential map $\exp\colon \mathfrak{heis} \to \mathrm{Heis}$ is surjective.

This group is called the Heisenberg group because the Lie bracket on its Lie algebra is $[X, Y] = Z$ for a suitable basis, which is reminiscent of the commutation relation between position and momentum in quantum mechanics.

Exercise 8

(a) Show that the Lie algebra of $\mathrm{GL}(n, \mathbb{R})$ is $\mathfrak{gl}(n, \mathbb{R})$.

(b) Show that the Lie algebra of $\mathrm{U}(n)$ is $\mathfrak{u}(n)$.

(c) Show that the Lie algebra of $\mathrm{SU}(n)$ is $\mathfrak{su}(n)$.

(d) Show that the Lie algebra of $\mathrm{O}(n)$ is $\mathfrak{o}(n)$.

(e) Show that the Lie algebra of $\mathrm{SO}(n)$ is $\mathfrak{so}(n)$.

Exercise 9

Let $A \in M_n(\mathbb{C})$. Then, the Hilbert-Schmidt norm is defined by

$$\|A\|_{HS} := \sqrt{\operatorname{tr}(A^\dagger A)}. \tag{1}$$

(a) Show that under the standard identification

$$M_n(\mathbb{C}) \to \mathbb{C}^{n^2}$$

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \mapsto \begin{pmatrix} a_{11} \\ \vdots \\ a_{1n} \\ \vdots \\ a_{n1} \\ \vdots \\ a_{nn} \end{pmatrix},$$

the Hilbert-Schmidt norm corresponds to the standard norm on $\mathbb{C}^n$.

(b) Show that the Hilbert-Schmidt norm satisfies

$$\|AB\|_{HS} \leq \|A\|_{HS} \|B\|_{HS}.$$

Exercise 10

Recall that the exponential function $\exp \colon \mathbb{R} \to \mathbb{R}$ is defined by the series

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

The goal of this exercise is to define the exponential map for a "Banach algebra" A. This is a real/complex vector space together with a product and a norm such that as a normed vector space A is complete (every Cauchy-sequence converges) and such that for all $a, b \in A$ we have $\|ab\| \leq \|a\|\|b\|$.

(a) Let $a \in A$ and $n \in \mathbb{N}$. Show that $\|a^n\| \leq \|a\|^n$.

(b) Consider now the sequence $\left(\frac{a^n}{n!}\right)_{n=0}^{\infty}$. Show that the partial sums satisfy

$$\left\| \sum_{k=0}^{n} \frac{a^n}{n!} \right\| \leq \sum_{k=0}^{n} \frac{\|a\|^n}{n!}, \tag{2}$$

for all $n \in \mathbb{N}$.

(c) Deduce from equation 2 that the series $\sum_{n=0}^{\infty} \frac{a^n}{n!}$ is absolutely convergent.

(d) Conclude that the map $\exp\colon A \to A$ defined by $\exp(a) = \sum_{n=0}^{\infty} \frac{a^n}{n!}$ is a well-defined, continuously differentiable map.

(e) Similarly, prove that $\log(x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(x-1)^k}{k}$ defines a well-defined continuous map in case $\|a\| < 1$.

Exercise 11

Let $(\mathfrak{g}, [\,\cdot\,,\,\cdot\,])$ be a Lie algebra. Let us now fix a basis $\{e_i\}$ of $\mathfrak{g}$. The structure coefficients of $\mathfrak{g}$ are defined by

$$[e_i, e_j] = \sum_k c_{ij}^k e_k.$$

(a) Show that $\mathfrak{g}$ is a Lie algebra if and only if its structure coefficients satisfy

  (i) for all $i, j, k \in \{1, \dots, n\}$, $c_{ij}^k = -c_{ji}^k$;

  (ii) for all $i, j, k, m \in \{1, \cdots, n\}$ $c_{ij}^l c_{lk}^m + c_{jk}^l c_{li}^m + c_{ki}^l c_{lj}^m = 0$.

(b) Compute the structure coefficients of $\mathfrak{gl}(n, \mathbb{C})$ with respect to the standard basis.

Exercise 12

Let $G \subseteq \mathrm{GL}(n, \mathbb{C})$ be a matrix Lie group and let $X \in \mathfrak{gl}(n, \mathbb{C})$.

(a) Suppose that $\exp(tX) \in G$ for $a < t < b$. Show that $\exp(tX) \in G$ for all $t \in \mathbb{R}$.

(b) Suppose that $\{t \in \mathbb{R} \mid \exp(tX) \in G\}$ admits an accumulation point. Show that $\exp(tX) \in G$ for all $t \in \mathbb{R}$.

3 Part 2

In this part we will answer question 1.1 posed in the beginning. To do this we need to take several steps which are more analytical in nature. We will first study products of exponentials and express them as a single exponential. After that we will consider subgroups of matrix Lie groups and we will establish a relation between Lie subalgebras and Lie subgroups. In the final section we will answer the posed question.

3.1 Baker-Campbell-Hausdorff

Our goal in this section is to write $e^X e^Y$ as a single exponential e^Z. In general this is not possible (if it were, the exponential map of any connected Lie group would be surjective, but this is false; see exercise 15), but one can always write products which are sufficiently close to the identity as the exponential of a single matrix, by virtue of theorem 2.27. In this section, we will give an explicit formula for $Z = \log(e^X e^Y)$.

We define an auxiliary function $g\colon B_1(1) \subset \mathbb{C} \to \mathbb{C}$ by

$$g(z) = \frac{\log z}{1 - z^{-1}}. \tag{3}$$

This function is holomorphic since we can express it as a power series which is absolutely convergent on the given disk:

$$g(z) = 1 + \sum_{m=1}^{\infty} (-1)^{m+1} \frac{(z-1)^m}{m(m+1)}.$$

The theorem we will prove is:

Theorem 3.1 (Baker-Campbell-Hausdorff; Integral form)**.** *There exists an $\varepsilon > 0$ such that for all $n \times n$ matrices X, Y such that $\|X\|, \|Y\| < \varepsilon$, we have*

$$\log\left(e^X e^Y\right) = X + \int_0^1 g\left(e^{\operatorname{ad}_X} e^{t \operatorname{ad}_Y}\right)(Y)\,\mathrm{d}t. \tag{4}$$

For the proof we need the following lemma. We want to apply a certain function to the operator ad_X in this lemma. Similarly, to the exponential function and the logarithm, if such a function is given by a power series, then we can use this power series to formally define the function on such an operator. We have

$$\frac{1 - e^{-z}}{z} = \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{(n+1)!},$$

which converges for all z. Then we say

$$\frac{1 - e^{-\operatorname{ad}_X}}{\operatorname{ad}_X} = \sum_{n=0}^{\infty} (-1)^n \frac{\operatorname{ad}_X^n}{(n+1)!}.$$

Lemma 3.2. *Let $J \subset \mathbb{R}$ be an interval and let $X\colon J \to M_n(\mathbb{C})$ be a smooth matrix-valued function. Then*

$$\frac{d}{dt} e^{X(t)} = e^{X(t)} \left(\frac{I - e^{-\operatorname{ad}_{X(t)}}}{\operatorname{ad}_{X(t)}} \left(\frac{dX}{dt} \right) \right).$$

Proof. See exercise 16. □

Remark 3.3. If we put $X(t) = tY$, then we obtain

$$\left.\frac{d}{dt} e^{tX}\right|_{t=0} = \sum_{n=0}^{\infty} (-1)^n \frac{(\operatorname{ad}_0)^n}{(n+1)!}(Y) = Y,$$

as we would expect. ♡

We also need the following equation.

Lemma 3.4. *Let $A \in \mathrm{GL}(n, \mathbb{C})$. Denote by $\mathrm{Ad}_A : \mathfrak{gl}(n, \mathbb{C}) \to \mathfrak{gl}(n, \mathbb{C})$ the map $X \mapsto AXA^{-1}$. Then we have*

$$\mathrm{Ad}_{e^X} = e^{\operatorname{ad}_X}.$$

Proof. Let $A(t) = \mathrm{Ad}_{e^{tX}}$ and $B(t) = e^{t\,\mathrm{ad}_X}$ for all $t \in \mathbb{R}$. Let $Y \in \mathfrak{gl}(n, \mathbb{C})$. Then we have

$$\begin{aligned}\frac{d}{dt}A(t)(Y) = \frac{d}{dt}\mathrm{Ad}_{e^{tX}}(Y) = \frac{d}{dt}e^{tX}Ye^{-tX} &= e^{tX}XYe^{-tX} - e^{tX}YXe^{-tX}\\ &= e^{tX}[X,Y]e^{-tX} = \mathrm{Ad}_{e^{tX}}(\mathrm{ad}_X(Y)) = A(t)\,\mathrm{ad}_X(Y).\end{aligned}$$

and $A(0) = \mathrm{Ad}_I = I$. On the other hand we have

$$\frac{d}{dt}B(t)(Y) = \frac{d}{dt}e^{t\,\mathrm{ad}_X}(Y) = e^{t\,\mathrm{ad}_X}\,\mathrm{ad}_X(Y) = B(t)\,\mathrm{ad}_X(Y).$$

and $B(0) = I$. So both satisfy the same ODE, so by the uniqueness of solutions of ODEs, we conclude that $A(t) = B(t)$ for all $t \in \mathbb{R}$, so in particular

$$\mathrm{Ad}_{e^X} = A(1) = B(1) = e^{\mathrm{ad}_X}.$$

□

We are now ready to prove the main theorem of this section.

Proof of theorem 3.1. Let $Z(t) = \log(e^X e^{tY})$. Then we have

$$\frac{d}{dt}e^{Z(t)} = e^X e^{tY} Y \quad \Rightarrow \quad e^{-Z(t)}\frac{d}{dt}e^{Z(t)} = Y. \tag{5}$$

On the other hand, lemma 3.2 gives

$$e^{-Z(t)}\frac{d}{dt}e^{Z(t)} = \left(\frac{I - e^{\mathrm{ad}_{Z(t)}}}{\mathrm{ad}_{Z(t)}}\right)\left(\frac{dZ}{dt}\right) = Y.$$

Now, for $Z(t)$ small enough, $\frac{I-e^{\mathrm{ad}_{Z(t)}}}{\mathrm{ad}_{Z(t)}}$ is invertible, since the differential of $A \mapsto \frac{I-e^A}{A}$ at $A = 0$ is invertible. Hence, we obtain an ODE for $Z(t)$

$$\frac{dZ}{dt} = \left(\frac{I - e^{\mathrm{ad}_{Z(t)}}}{\mathrm{ad}_{Z(t)}}\right)^{-1}(Y). \tag{6}$$

The map Ad in lemma 3.4 satisfies

$$\mathrm{Ad}_{e^{Z(t)}} = \mathrm{Ad}_{e^X e^{tY}} = \mathrm{Ad}_{e^X}\,\mathrm{Ad}_{e^{tY}},$$

so by lemma 3.4

$$e^{\mathrm{ad}_{Z(t)}} = e^{\mathrm{ad}_X}e^{t\,\mathrm{ad}_Y}$$

and so $\mathrm{ad}_{Z(t)} = \log(e^{\mathrm{ad}_X}e^{\mathrm{ad}_{tY}})$. Plugging this into equation 6 we obtain

$$\frac{dZ}{dt} = \left(\frac{I - \left(e^{\mathrm{ad}_X}e^{\mathrm{ad}_{tY}}\right)^{-1}}{\log\left(e^{\mathrm{ad}_X}e^{\mathrm{ad}_{tY}}\right)}\right)^{-1}(Y) = g\left(e^{\mathrm{ad}_X}e^{\mathrm{ad}_{tY}}\right)(Y).$$

So by the fundamental theorem of calculus we conclude

$$\log(e^X e^Y) = Z(1) = X + \int_0^1 g\left(e^{\mathrm{ad}_X}e^{\mathrm{ad}_{tY}}\right)(Y)\,\mathrm{d}t,$$

where we used that $Z(0) = X$. □

Remark 3.5. As we can see from the Baker-Campbell-Hausdorff formula, the product of the Lie group and the Lie bracket are tightly related, as one would expect. Using the Taylor series for the logarithm,

$$\log z = \sum_{m=1}^{\infty} (-1)^{m+1} \frac{(z-1)^m}{m},$$

we can write

$$g(z) = 1 + \frac{1}{2}(z-1) + \text{h.o.t.}$$

Similary, we can expand

$$e^{\mathrm{ad}_X} e^{t\,\mathrm{ad}_Y} - I = \mathrm{ad}_X + t\,\mathrm{ad}_Y + \text{h.o.t.}$$

Combining this with the Baker-Campbell-Hausdorff formula, we obtain

$$\log(e^X e^Y) = X + Y + \frac{1}{2}[X, Y] + \text{h.o.t.}$$

This is the series expansion of the Baker-Campbell-Hausdorff formula. See exercise 14 for more about the series form. ♡

3.2 Lie subgroups and subalgebras

Definition 3.6. If G is a matrix Lie group with Lie algebra $\mathfrak{g}$, then $H \subset G$ is a *connected Lie subgroup* of G if the following conditions are satisfied:

(i) H is a subgroup of G.
(ii) The Lie algebra $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$.
(iii) Every element of H can be written in the form $e^{X_1} e^{X_2} \cdots e^{X_m}$, with $X_1, \dots, X_m \in \mathfrak{h}$.

♢

Example 3.7. Consider the 2-torus $T^2 \subset \mathrm{GL}(2, \mathbb{C})$. Let $\alpha \in \mathbb{R}$. Then

$$H_\alpha = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{i\alpha\theta} \end{pmatrix} \in T^2 \middle| \theta \in \mathbb{R} \right\}$$

is a connected Lie subgroup of T^2. If $\alpha \in \mathbb{Q}$, then this subgroup is isomorphic to S^1. If $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, then it is isomorphic to $(\mathbb{R}, +)$. In the latter case $H_\alpha \subset T^2$ is dense. △

Theorem 3.8. *Let G be a matrix Lie group with Lie algebra $\mathfrak{g}$ and let $\mathfrak{h} \subset \mathfrak{g}$ be a Lie subalgebra. Then there exists a unique connected Lie subgroup H of G with Lie algebra $\mathfrak{h}$.*

Writing down what should be the connected Lie subgroup H is easy:

$$H = \left\{ e^{X_1} e^{X_2} \cdots e^{X_k} \middle| X_1, \dots, X_k \in \mathfrak{h} \right\}.$$

However, there is no *a priori* guarantee that the Lie algebra of H will be $\mathfrak{h}$. To show this, we will use that $\mathfrak{h}$ is a Lie algebra and the Baker-Campbell-Hausdorff formula. To see what may fail if $\mathfrak{h}$ is not a Lie algebra, consider the following example.

Example 3.9. Consider the Heisenberg group and Lie algebra from exercise 7. Let

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{heis}.$$

Then $[X, Y] = Z$ and for their exponentials we have

$$\exp(Z) = \exp(-X)\exp(-Y)\exp(X)\exp(Y).$$

So with $\mathfrak{h} = \langle X, Y \rangle$ and H as above we have $H = \mathrm{Heis}$. △

Pick a complement N of $\mathfrak{h}$, so that $\mathfrak{g} = \mathfrak{h} \oplus N$. Since the exponential map is a local diffeomorphism around $0 \in \mathfrak{g}$, we have open neighbourhoods $U \subset \mathfrak{h}$ and $V \subset N$ of 0 such that $f\colon U \times V \to f(U \times V), (X, Y) \mapsto \exp(X)\exp(Y)$ is a diffeomorphism. We clearly have $\mathfrak{h} \subset \mathrm{Lie}(H)$, so we must show that $\mathrm{Lie}(H) \subset \mathfrak{h}$.

Suppose that $Y \in \mathrm{Lie}(H)$, then $\exp(tY) \in H$ for all $t \in \mathbb{R}$ and so we must show that $\frac{d}{dt}\big|_{t=0} \exp(tY) \in \mathfrak{h}$ based on this assumption. Since differentiation is local, we may assume that t is small such that $\exp(tY) \in f(U \times V)$. So then we can write $\exp(tY) = \exp(X(t))\exp(Z(t))$ with $X(t) \in \mathfrak{h}$ and $Z(t) \in N$ and both depending smoothly on t. Then we have $\frac{d}{dt}\big|_{t=0} \exp(tY) = X'(0) + Z'(0)$, so we want to show that $Z'(0) = 0$. The problem here is that for elements arbitrarily close to the identity of H we may have that $Z(t) \neq 0$, see example 3.7. However, the situation turns out to be not too bad.

Lemma 3.10. *The set $E = \{Z \in V |\, e^Z \in H\}$ is countable.*

To prove this lemma we need the notion of rational elements of $\mathfrak{h}$.

Definition 3.11. Let $\mathfrak{h}$ be a Lie algebra and fix a basis. A *rational element* of $\mathfrak{h}$ is an element with rational coefficients with respect to the chosen basis. ◇

Remark 3.12. If $\mathfrak{h}$ is n-dimensional, then the set of rational elements with respect to a given basis is isomorphic to $\mathbb{Q}^n$ and therefore there are countably many rational elements. ♡

The significance of rational elements is that they are dense in the Lie algebra. The proof of this lemma uses the Baker-Campbell-Hausdorff formula.

Lemma 3.13. *Let $\mathfrak{h}$ be a Lie algebra with a fixed basis. Let $\delta > 0$ and $A \in H$. Then there are rational elements $R_1, \dots, R_m \in \mathfrak{h}$ and an element $X \in \mathfrak{h}$ such that*

$$A = e^{R_1} \cdots e^{R_m} e^X,$$

and $\|X\| < \delta$.

Proof. Denote the right-hand side of the Baker-Campbell-Hausdorff formula, equation 4, by $C(X, Y)$. Then there exists some $\varepsilon > 0$ such that for $\|X\|, \|Y\| < \varepsilon$ we have $e^X e^Y = e^{C(X,Y)}$. This map C is continuous in X and Y. So assume that $\delta < \varepsilon$ and that $\|X\|, \|Y\| < \delta$ implies that $\|C(X, Y)\| < \varepsilon$.

Given e^X, we can write it as $(e^{X/k})^k$ so that we write

$$A = e^{X_1} \cdots e^{X_N},$$

with all $\|X_i\| < \delta$. We proceed by induction on N. If $N = 1$, then we don't need to show anything. Now, assume that we have show the result for $N \geq 1$ and consider

$$A = e^{X_1} \cdots e^{X_N} e^{X_{N+1}} = e^{R_1} \cdots e^{R_m} e^{X} e^{X_{N+1}} = e^{R_1} \cdots e^{R_m} e^{C(X, X_{N+1})},$$

where we have applied the induction hypothesis in the middle equality. Then $\|C(X, X_{N+1})\| < \varepsilon$. Since rational elements are dense in $\mathfrak{h}$ we may choose an rational element R_{m+1} arbitrarily close to $C(X, X_{N+1})$ such that

$$\|C(-R_{m+1}, C(X, X_{N+1}))\| < \delta.$$

This is possible, since we have $C(-Z, Z) = \log(e^{-Z}e^{Z}) = 0$ and by density, we can find a sequence of rational elements $R_{m+1}^{j} \to C(X, X_{N+1})$ as $j \to \infty$, so $C(-R_{m+1}^{j}, C(X, X_{N+1})) \to 0$ as $j \to \infty$.

Then we have

$$A = e^{R_1} \cdots e^{R_m} e^{R_{m+1}} e^{-R_{m+1}} e^{C(X, X_{N+1})} = e^{R_1} \cdots e^{R_{m+1}} e^{X'},$$

where $X' = C(-R_{m+1}, C(X, X_{N+1}))$. Then we are done. □

This lemma implies the countability of the set above.

Proof of lemma 3.10. For $A \in f(U \times V)$ one can write $A = e^{X}e^{Z}$ with $X \in U$ and $Z \in V$ uniquely. Let δ be small enough such that for $\|X\|, \|Y\| < \delta$ we have that $X, Y \in U$, and $C(X, Y)$ exists and is contained in U.

Suppose that

$$e^{Z_1} = e^{R_1} \cdots e^{R_m} e^{X_1} \quad \text{and} \quad e^{Z_2} = e^{R_1} \cdots e^{R_m} e^{X_2},$$

both in $\exp(V)$. Then $e^{-Z_1} = e^{-X_1}e^{X_2}e^{-Z_2} = e^{C(-X_1, X_2)}e^{-Z_2}$. However $\exp(V) \subset f(U \times V)$, so by the bijectivity of f it follows that $Z_1 = Z_2$. But then $e^{X_1} = e^{X_2}$, and so again by bijectivity of f, we have $X_1 = X_2$.

So for each finite sequence of rational elements there is at most one X with norm less than δ such that $e^{R_1} \cdots e^{R_m} e^{X} \in \exp(V)$. By the previous lemma each element of H can be expressed in this form, so $E = \exp(V) \cap H$ is countable. □

Now, the proof of our theorem boils down to a simple continuity argument.

Proof of theorem 3.8. For $Y \in \mathrm{Lie}(H)$ we may write $e^{tY} = e^{X(t)}e^{Z(t)}$ for t small, where $X(t)$ and $Z(t)$ are smooth. We know that $Z(0) = 0$ since $e^0 = I \in H$ and f is bijective. Suppose that $Z(t)$ is not constant. Then Z assumes uncountably many values by continuity. But by assumption $e^{tY} \in H$ for all t and $e^{X(t)} \in H$ for all t by construction, so $e^{Z(t)} \in H$ for all t as well. But then $Z(t) \in E$ for all t, which is a countable set. We have reached a contradiction and so $Z(t)$ must be constant and therefore $Y = X'(0) \in \mathfrak{h}$. □

3.3 Lie's third theorem

In this section we will answer the question posed in the introduction. It turns out that this question has a positive answer. However, the proof presented here relies on a deep result from the representation theory of Lie algebras, which we will not prove here.

Looking at theorem 3.8, we can split our original question 1.1 into two questions.

Question 3.14. Is every Lie algebra isomorphic to a real Lie subalgebra of some $\mathfrak{gl}(n, \mathbb{C})$?

Question 3.15. Is every connected Lie subgroup a matrix Lie group?

These turn out to be two difficult questions, for which we will unfortunately not be able to provide full proofs to their answers.

To start with the first question, this is answered positively and this is a classical result in Lie algebra theory:

Theorem 3.16 (Ado's theorem)**.** *Every finite-dimensional real Lie algebra* $\mathfrak{g}$ *can be identified with a real Lie subalgebra of* $\mathfrak{gl}(n, \mathbb{C})$ *for a sufficiently large* n*.*

The proof of this theorem can be found in appendix E of [FH04]. Using this result we can prove the following theorem.

Theorem 3.17. *Let* $\mathfrak{g}$ *be a finite-dimensional real Lie algebra. Then there exists a connected Lie subgroup* G *with Lie algebra* $\mathfrak{g}$*.*

Proof. By Ado's theorem, we can identify $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{C})$. Then by theorem 3.8, there is a connected Lie subgroup $G \subset \mathrm{GL}(n, \mathbb{C})$ such that $\mathrm{Lie}(G) = \mathfrak{g}$. $\square$

However, we still have to answer the second question, namely whether every connected Lie subgroup is a matrix Lie group (i.e. a closed subgroup of some $\mathrm{GL}(n, \mathbb{C})$). This turns out to have a positive answer as well. The following theorem was proven by M. Goto in [Got50].

Theorem 3.18. *Every connected Lie subgroup of* $\mathrm{GL}(n, \mathbb{C})$ *is isomorphic to a closed subgroup of* $\mathrm{GL}(d, \mathbb{C})$ *for some integer* d*.*

Note that the integer d does not have to equal n. This allows us to finally answer the original question we posed in the introduction.

Theorem 3.19 (Lie's third theorem)**.** *Let* $\mathfrak{g}$ *be a finite-dimensional real Lie algebra. Then there exists a matrix Lie group* G *such that* $\mathrm{Lie}(G) = \mathfrak{g}$*.*

Proof. By theorem 3.17 there is a connected Lie subgroup $G \subset \mathrm{GL}(n, \mathbb{C})$ with the given Lie algebra $\mathfrak{g}$. Then by theorem 3.18, this G is isomorphic to a closed subgroup of some $\mathrm{GL}(d, \mathbb{C})$, so G is a matrix Lie group. $\square$

3.4 Exercises

Exercise 13

Show that all connected Lie subgroups of the 2-torus T^2 are: $\{I\}, H_\alpha, H_\infty, T^2$, with H_α from example 3.7 and

$$H_\infty = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix} \middle| \theta \in \mathbb{R} \right\}$$

Hint: Use theorem 3.8.

Exercise 14

In this exercise, we further study the series form of the Baker-Campbell-Hausdorff formula.

(a) Show that the power series of $g(z)$ in equation (3) is given by

$$g(z) = 1 + \sum_{m=1}^{\infty} (-1)^{m+1} \frac{(z-1)^m}{m(m+1)}.$$

(b) Show that the following equation holds

$$e^{\operatorname{ad}_X} e^{t\operatorname{ad}_Y} - I = \sum_{m=1}^{\infty} \sum_{k=0}^{m} \frac{\operatorname{ad}_X^{m-k} \operatorname{ad}_Y^k}{k!(m-k)!} t^k.$$

(c) Expand $g(e^{\operatorname{ad}_X} e^{t\operatorname{ad}_Y})$ up to order 2 (at most 1 composition of the adjoint maps).

(d) Apply this expansion to Y and use the Baker-Campbell-Hausdorff formula to show that

$$\log(e^X e^Y) = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}\left([X,[X,Y]] - [Y,[X,Y]]\right) + \text{h.o.t.}$$

Exercise 15

Consider the matrix

$$A = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}.$$

(a) Show that $A \in \mathrm{SL}_2(\mathbb{C})$.

(b) Show that A is not diagonalizable.

(c) Show that an $X \in \mathfrak{sl}_2(\mathbb{C})$ with distinct eigenvalues is diagonalizable.

(d) Show that if $X \in \mathfrak{sl}_2(\mathbb{C})$ is diagonalizable, then so is e^X.

(e) Show that the only $X \in \mathfrak{sl}_2(\mathbb{C})$ with repeated eigenvalues is 0.

(f) Conclude that there is no $X \in \mathfrak{sl}_2(\mathbb{C})$ such that $e^X = A$.

(g) Give an example that show that exp is not always injective.

(h) Give an example where exp defines a bijection between a Lie group G and its Lie algebra $\mathfrak{g}$.

Exercise 16

The goal of this exercise is to prove that for all $X, Y \in M_n(\mathbb{C})$ we have

$$\frac{d}{dt} e^{X+tY}\bigg|_{t=0} = e^X \left(\frac{I - e^{-\operatorname{ad}_X}}{\operatorname{ad}_X}(Y) \right).$$

The expression on the right-hand side comes from the function

$$\frac{1-e^z}{z} = \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{(n+1)!}. \tag{7}$$

(a) Use d'Alembert's ratio test to show that the series in equation 7 converges for all $z \in \mathbb{C}$.

(b) For a fixed $x \in \mathbb{R}$ compute the integral of $t \mapsto e^{-tx}$ on $[0, 1]$.

(c) Conclude from this

$$\frac{1-e^{-Z}}{Z} = \int_0^1 e^{-tZ}\, \mathrm{d}t$$

for $Z \in M_n(\mathbb{C})$

(d) Argue that the Riemann sum corresponding to a uniform subdivision in m subintervals of the function $t \mapsto e^{-tZ}$ is given by

$$\frac{1}{m} \sum_{k=0}^{m-1} \left(e^{-Z/m}\right)^k$$

(e) Conclude from items (c) and (d) that

$$\lim_{m\to\infty} \frac{1}{m} \sum_{k=0}^{m-1} \left(e^{-Z/m}\right)^k = \frac{1-e^{-Z}}{Z}. \tag{8}$$

(f) For $X, Y \in M_n(\mathbb{C})$, define $\Delta(X, Y) = \frac{d}{dt} e^{X+tY}\big|_{t=0}$. Compute $\Delta(0, Y)$.

(g) Write $e^{X+tY} = \left(\exp\left(\frac{X}{m} + t\frac{Y}{m}\right)\right)^m$ for $m \geq 1$. Use the product rule to show

$$\frac{d}{dt} e^{X+tY}\bigg|_{t=0} = e^{\frac{(m-1)X}{m}} \sum_{k=0}^{m-1} \left(e^{X/m}\right)^{-k} \Delta\left(\frac{X}{m}, \frac{Y}{m}\right) \left(e^{X/m}\right)^k.$$

(h) Use lemma 3.4 to conclude

$$\frac{d}{dt} e^{X+tY}\bigg|_{t=0} = e^{\frac{(m-1)X}{m}} \sum_{k=0}^{m-1} \exp\left(-\frac{\operatorname{ad}_X}{m}\right)^k \left(\Delta\left(\frac{X}{m}, \frac{Y}{m}\right)\right). \tag{9}$$

(i) Take the limit $m \to \infty$ in equation 9 and use equation 8 to conclude

$$\frac{d}{dt} e^{X+tY}\bigg|_{t=0} = e^X \left(\frac{I - e^{-\operatorname{ad}_X}}{\operatorname{ad}_X}(Y) \right).$$

(j) Use the chain rule to show that for a smooth matrix-valued function $X(t)$ one has

$$\frac{d}{dt} e^{X(t)} = e^{X(t)} \left(\frac{I - e^{-\operatorname{ad}_{X(t)}}}{\operatorname{ad}_{X(t)}} \left(\frac{dX}{dt}\right) \right).$$

4 Outlook

We hope that this has been an enjoyable introduction into the world of Lie groups. Of course, we have only scratched the surface of what the theory has to offer. For example, we discussed Lie's *third* theorem, which begs the question: Are there also a first and second theorem? The answer is yes. The first theorem says that two Lie groups with isomorphic Lie algebras are locally isomorphic (one can deduce this from the Baker-Campbell-Hausdorff formula). The second theorem says that under suitable topological conditions, a morphism of Lie algebras gives a Lie group morphism. In fact, the Lie's three theorem together say that there is a one-to-one correspondence between finite-dimensional real Lie algebras and Lie groups subject to certain topological conditions.

If you want to learn more about this topic, our main reference [Hal15] is a good place to start. This treats for a large part the representation theory of Lie groups and algebras, which is a rich theory of its own with applications in quantum mechanics, for example.

To study Lie groups in general, one needs to know what a smooth manifold is. For this, there is a large body of literature. For example, [Lee13] is a comprehensive book that treats smooth manifolds in great detail. An alternative and shorter book is [War83]. A book treating Lie groups in general is, for example, [DK00].

If you want to learn more about the analysis of this story, you can pick up an advanced real analysis book like [DKB04; Gar13]. After that, one can learn some functional analysis. This leads (among other things) to functional calculus, which provides a general framework for applying continuous/holomorphic functions to operators. Some references for this are [Mac09; Con10].

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