Lecture notes Donaldson seminar: Construction and local model Yang-Mills moduli space

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1 ADHM construction

The following is based on Chapter 3 of [1].

1.1 General theory

The ADHM construction relates the existence of ASD connections over the 4-sphere S^4 to algebraic data. This data depends on two integers, n and k representing the rank of the $SU(n)$-bundle and the second Chern class of the bundle, respectively. We have for a connection A on $\mathbb{R}^4$ of finite-energy the following equation

$$\kappa(A) = \frac{1}{8\pi^2}\int_{\mathbb{R}^4} |F_a|^2 d\mu \overset{\text{ASD}}{=} \frac{1}{8\pi^2}\int \text{Tr}(F_a^2) d\mu = c_2(E) \in \mathbb{Z}.$$

We have the following data:

1. A k-dimensional complex vector space $\mathcal{H}$ with a Hermitian metric.
2. An n-dimensional complex vector space E_∞ with Hermitian metric and determinant form.
3. Self-adjoint linear maps $T_i : \mathcal{H} \to \mathcal{H}$, $i = 1, 2, 3, 4$.
4. A linear map $P : E_\infty \to \mathcal{H} \otimes S^+$, where S^+ is the two-dimensional positive spin space of V.

From this data we can construct some other maps. First of all, for each $x \in \mathbb{R}^4$ we define

$$R_x : \mathcal{H} \otimes S^- \oplus E_\infty \to \mathcal{H} \otimes S^+,$$

by

$$R_x = \left(\sum_{i=1}^4 (T_i - x_i 1) \otimes \gamma(e_i)^*\right) \oplus P$$

Secondly, we can use the Hermitian structures on E_∞ and $\mathcal{H} \otimes S^+$ to construct the adjoint of P. Then the composition $PP^* \in \text{End}(\mathcal{H} \otimes S^+)$. We have $\text{End}(\mathcal{H} \otimes S^+) = \text{End}(\mathcal{H}) \otimes \text{End}(S^+)$. Then we have the action of self-dual forms on the positive spinors from last week,

$$\Lambda^+ \cong \mathfrak{su}(S^+) \subset \text{End}(S^+),$$

and so we can consider the component of PP^* in $\text{End}(\mathcal{H}) \otimes \Lambda^+$. We have three self-dual 2-forms that form a basis, which we call I, J, K, and so we get

$$PP_I^*, PP_J^*, PP_K^* \in \text{End}(\mathcal{H}).$$

Definition 1.1. A system of *ADHM data* for group $SU(n)$ and index k, is a system $(\mathcal{H}, E_\infty, T, P)$ as above, which satisfy:

1. The ADHM equations:

$$[T_1, T_2] + [T_3, T_4] = PP_I^*$$
$$[T_1, T_3] + [T_4, T_2] = PP_J^*$$
$$[T_1, T_4] + [T_2, T_3] = PP_K^*$$

2. The non-degeneracy conditions: For each $x \in \mathbb{R}^4$ the map R_x is surjective.

The non-degeneracy condition ensures that the maps R_x are a surjective vector bundle morphism, hence they cut out a kernel bundle. It is on this kernel bundle that we get a connection by the same projection trick as last week. We denote this connection by $A(T, P)$.

We have the following notion of equivalence: Picking bases for $\mathcal{H}$ and E_∞ we can view the T_i and P as matrices. Then the systems (T, P) and (T', P') are equivalent if there exist $g \in U(k)$ and $h \in SU(n)$ such that

$$T_i' = gT_ig^{-1}, \quad P' = gPh^{-1}.$$

Then the central result is

Theorem 1.2. *The map $(T, P) \mapsto A(T, P)$ gives a one-to-one correspondence between equivalence classes of ADHM data for group $SU(n)$ and index k, and gauge equivalence classes of finite energy, ASD $SU(n)$-connections A over $\mathbb{R}^4$ with $\kappa(A) = k$.*

Then by a theorem of Uhlenbeck these finite-energy ASD connections extend over "infinity" and give an ASD connection on S^4.

1.2 Explicit example

We now focus on the simplest example of the ADHM construction: $SU(2)$ solutions with Chern class $c_2 = 1$. In this case $\mathcal{H}$ is 1-dimensional, and so the maps T_i are just scalars. The map P is an element of $\mathrm{Hom}(E_\infty, S^+)$. Since the $[T_i, T_j] = 0$ for all $1 \le i, j \le 4$, we see that the ADHM equations say that $(PP^*)_{\Lambda^+} = 0$, so the trace-free part of PP^* is zero (the trace-free part is the component in $\mathfrak{su}(S^+)$). Therefore, $PP^* \in \mathbb{C}\mathrm{Id}$. Hence P mod $U(1) \times SU(2)$ is classified by its Frobenius norm $|P|$, which is by definition $\sqrt{\mathrm{tr}(PP^*)}$. Namely, we have $P = \lambda U$ for some isometry U and we can absorb $\det U = -1$ into the $U(1)$ action.

The non-degeneracy condition implies in particular that R_x is surjective for $x_i = T_i$. There we have

$$R_x = 0 \oplus P,$$

hence we require P to be surjective. Hence $|P| > 0$ and vice versa, if $|P| > 0$, then we have P surjective. Therefore, the solutions are classified by $\mathbb{R}^4 \times (0, \infty)$.

In fact, we can identify the $\mathbb{R}^4$ in this moduli space with the coordinate plane $\mathbb{R}^4$, by considering $x_i = T_i$ the "center of mass" of the solution. Indeed, looking at R_x and translating the coordinates by a vector $w \in \mathbb{R}^4$, we see that we should T_i by $T_i + w_i 1$ to keep the same connection in these translated coordinates. Similarly, a dilation of the coordinates by $t > 0$ forces us to replace the T_i by tT_i and P by tP. So translations and dilations act transitively on the moduli space, and so if we have one solution, then we have all. One can write down explicit expressions for this "basic instanton" [1, p. 116].

2 General construction of the moduli space

The material is a cross-section of the material of Chapter 4 in [1].

The goal is to study the moduli space of ASD connections on Hermitian vector bundle over a compact 4-manifold. Intuitively, we would like to define this as follows. Let $\mathcal{C}$ be the affine space of all smooth connections, and let $\mathcal{S} \subset \mathcal{C}$ be the subset of connections $A \in \mathcal{C}$ such that it is anti-self-dual, i.e. $F_A^+ = 0$. We also have a group $\mathcal{G}$ of smooth gauge transformations acting on $\mathcal{S}$, and we would like our moduli space to be $\mathcal{M} = \mathcal{S}/\mathcal{G}$. The technical problem however is that spaces of smooth sections are poorly behaved from an analytic perspective (in the best case they would be Fréchet spaces). Therefore, we want to relax smoothness (temporarily) to construct the moduli space.

2.1 Sobolev spaces, Sobolev embeddings and Sobolev multiplication

Instead of working with spaces of smooth objects, we replace these by Sobolev spaces. The upshot is that these Sobolev spaces are locally Hilbert spaces and the global objects will in general be Hilbert manifolds. These Sobolev spaces depend on a real parameter $s \in \mathbb{R}$ and we denote these by H^s. We call this parameter s the regularity of the Sobolev space. I will not dive into the specifics, but I will make a few remarks about this. For more details on the underlying theory, I found Chapter 3 of [2] to be useful.

- **Hilbert manifolds:** Hilbert spaces are nice because we still have an inverse function theorem that only depends on the derivative at a point. As a motto, "most things that work for finite-dimensional manifolds, still work for Hilbert manifolds" (one notable exception is anything that relies on local compactness).
- **Sobolev embedding:** There is an embedding of Sobolev spaces into spaces of C^k-functions provided the "regularity" of the Sobolev space is high enough. In dimension four, we have $H^s \hookrightarrow C^k$ if $s > 2 + k$.
- **Sobolev multiplication:** We can multiply functions in Sobolev spaces, but this comes at a cost of regularity in general. That is, if we have $f \in H^{s_1}$ and $g \in H^{s_2}$ and B some smooth bilinear map, then for $0 \leq s \leq s_1, s_2$ and $s_1 + s_2 - s > 2$, we have $B \circ (f, g) \in H^s$. So if we have $s > 2$, then we get multiplication $H^s \times H^s \to H^s$.

It turns out that the moduli space does not depend on the choice of regularity $s > 2$, so we will suppress this in the notation.

2.2 Gauge quotient and reducible connections

Let $s > 2$. Let $\mathcal{A}$ denote the space of H^{s-1} connections and $\mathcal{G}$ the group of H^s gauge transformations, so that the gauge elements are at least continuous and acting on the connections without lowering the regularity. Instead of first cutting out the solution set in $\mathcal{A}$ and then taking the quotient, we work the other way around: We first look at the quotient $\mathcal{B} = \mathcal{A}/\mathcal{G}$. The L^2-norm on $\mathcal{A}$ descends to a metric on $\mathcal{B}$ (this is a nontrivial statement about the orbits of the action being closed). Since the Sobolev topology is finer than the L^2-topology, it follows that the quotient is Hausdorff.

Next, we want to understand the local structure of $\mathcal{B}$. Recall that the gauge action is given by [1, Eq. (2.1.8)]

$$u(A) = A - (\nabla_A u)u^{-1}$$

So if we differentiate this in the identity for fixed A, we get

$$\frac{d}{dt}|_0 \left(A - (\nabla_A u_t)u_t^{-1}\right) = -(\nabla_A \dot{u}_0)u_0 - (\nabla_A u_0)\frac{d}{dt}|_0(u_t^{-1}) = -\nabla_A \xi + (\nabla_A 1)\xi = -\nabla_A \xi,$$

where $\xi = \dot{u}_0$. So the differential of the action is given by $-d_A : \Omega^0(\mathfrak{g}_E) \to \Omega^1(\mathfrak{g}_E)$.

It turns out that d_A is part of an elliptic complex, and so we have the decomposition

$$\Omega^1(\mathfrak{g}_E) = \operatorname{im} d_A \oplus \ker d_A^*, \quad \Omega^0(\mathfrak{g}_E) = \ker d_A \oplus \operatorname{im} d_A^*.$$

For $A \in \mathcal{A}$ and $\varepsilon > 0$ we set

$$T_{A,\varepsilon} = \{a \in \Omega^1(\mathfrak{g}_E) \mid d_A^* a = 0, \|a\|_{H^{s-1}} < \varepsilon\}.$$

The neighbourhood of $[A] \in \mathcal{B}$ can be described as a quotient of $T_{A,\varepsilon}$, which follows from Coulomb gauge fixing (this is precisely solving the $d_A^* a = 0$ condition).

Next we consider the action of the gauge group on these slices. The stabilizer of a connect A is given by $\Gamma_A = \{u \in \mathcal{G} \mid u(A) = A\}$. This condition forces that $\nabla_A u = 0$, so the value at a point determines the value everywhere by parallel transport. Therefore we can view $\Gamma_A \subset \operatorname{Aut}(E_x) \cong G$. The stabilizer Γ_A acts on $T_{A,\varepsilon}$ by $u(a) = uau^{-1} = \operatorname{Ad}(u)a$, so we see that the center $Z(G)$ acts trivially.

Proposition 2.1. *For ε small enough the projection map from $\mathcal{A}$ to $\mathcal{B}$ induces a homeomorphism from the quotient $T_{A,\varepsilon}/\Gamma_A$ to a neighbourhood of $[A]$ in $\mathcal{B}$.*

By the observation above, we will always have $Z(G) \subset \Gamma_A$. The set for which $\Gamma_A = Z(\mathcal{G})$ we denote by $\mathcal{A}^*$ and the image in the quotient by $\mathcal{B}^*$. Since $Z(G)$ acts trivially, we get that $\mathcal{B}^*$ is locally homeomorphic to $T_{A,\varepsilon}$. The elements of $\mathcal{B}^*$ are customarily called *irreducible connections*.

2.3 Fredholm theory

Before discussing the local model for the moduli space, we will need a short interlude on Fredholm theory.

Definition 2.2. Let U, V be Banach spaces and $L : U \to V$ a bounded linear map. We say it is *Fredholm* if its kernel and cokernel are finite-dimensional. The *index* of L is defined as

$$\operatorname{ind}(L) = \dim \ker L - \dim \operatorname{coker} L.$$

The Fredholm assumption implies that we can write

$$U = U_0 \oplus F, \quad V = V_0 \oplus G$$

such that $L : U_0 \to V_0$ is an isomorphism between closed subspaces, and F, G are finite-dimensional.

We also have a non-linear version:

Definition 2.3. Let $N \subset U$ be a connected open neighbourhood of 0. A smooth map $\phi : N \to V$ is called *Fredholm* if for each point x in N the derivative $(D\phi)_x : U \to V$ is Fredholm. The *index* of ϕ is defined as the index of $(D\phi)_x$, which is independent of x.

We say that two functions ϕ, ψ are right-equivalent if there exists a local diffeomorphism f such that $\phi \circ f = \psi$. Then the central result we need is

Proposition 2.4. *A Fredholm map ϕ from a neighbourhood of 0 is locally right equivalent to a map of the form*

$$\tilde{\phi} : U_0 \times F \to V_0 \times G, \tilde{\phi}(\xi, \eta) = (L(\xi), \alpha(\xi, \eta))$$

where L is a linear isomorphism $U_0 \to V_0$, and F, G are finite-dimensional, $\dim F - \dim G = \operatorname{ind} \phi$*, and the derivative of α vanishes at* 0.

This gives us a finite-dimensional model for the zero-set of ϕ: It is locally diffeomorphic to the zero-set of $f : F \to G$ given by $y \mapsto \alpha(0, y)$.

2.4 Local model of the moduli space

The moduli space $\mathcal{M}$ is defined as the image of $\{A \in \mathcal{A} \mid F_A^+ = 0\}$ in $\mathcal{B}$.

If we restrict attention to the slice $T_{A,\varepsilon}$ with A ASD, then we can rewrite the condition using

$$F_{A+a} = F_A + d_A a + a \wedge a \Rightarrow F_{A+a}^+ = d_A^+ a + (a \wedge a)^+,$$

and the moduli space then corresponds to the zero set of the map

$$\psi : T_{A,\varepsilon} \to \Omega^+(\mathfrak{g}_E), a \mapsto F_{A+a}^+ = d_A^+ a + (a \wedge a)^+$$

modulo Γ_A.

The map ψ has derivative at 0 given by

$$d_A^+ : \ker d_A^* \to \Omega_X^+(\mathfrak{g}_E).$$

The operator $\delta_A = d_A^* \oplus d_A^+$ is elliptic, hence Fredholm. Therefore, the map ψ is Fredholm, as it is the restriction to a ball in $\ker d_A^*$. The index is given by

$$\operatorname{ind} \psi = \operatorname{ind} \delta_A + \underbrace{\dim \Gamma_A}_{\dim \ker d_A}.$$

The result from Fredholm theory gives

Proposition 2.5. *If A is an ASD connection over X, a neighbourhood of $[A]$ in $\mathcal{M}$ is modelled on a quotient $f^{-1}(0)/\Gamma_A$ where*

$$f : \ker \delta_A \to \operatorname{coker} d_A^+$$

is a Γ_A-equivariant map.

If 0 is a regular value, than we could use this to compute the actual dimension of the moduli space. In that case we would have dimension

$$s = \dim \ker \delta_A - \dim \operatorname{coker} d_A^+ - \dim \Gamma_A = \operatorname{ind} \psi - \dim \Gamma_A = \operatorname{ind} \delta_A.$$

For this reason, we call this number the "virtual dimension". We can compute this using the Atiyah-Singer index theorem (a more general version of Riemann-Roch-Hirzebruch). For a $SU(2)$-connection then the formula is

$$s = 8c_2(E) - 3(1 - b_1(X) + b_+(X)).$$

Going back the example at the start, we had $c_2 = 1$, and for the sphere we have $b_1 = b_+ = 0$, hence the expected dimension is indeed $s = 8 - 3 = 5$.

2.5 Deformation complex

Finally, we can take a more invariant perspective: We package the linearized picture into a "deformation complex":

$$\Omega_X^0(\mathfrak{g}_E) \xrightarrow{d_A} \Omega_X^1(\mathfrak{g}_E) \xrightarrow{d_A^+} \Omega_X^+(\mathfrak{g}_E)$$

The ASD condition for A implies $d_A^+ \circ d_A = 0$, so we have three cohomology groups coming from this complex: H_A^0, H_A^1, H_A^2. We can think of H_A^0 as the infinitesimal automorphisms, and H_A^2 as some obstructions for deformations. Finally, the middle cohomology is the linearization of the ASD equations modulo the gauge group. Then the "virtual dimension" is given by

$$s = \dim H_A^1 - \dim H_A^0 - \dim H_A^2 = -\chi,$$

minus the Euler characteristic of the complex. The local model is a map

$$f : H^1_A \longrightarrow H^2_A$$

which encodes the non-linear part in the equations, i.e. locally it is given by

$$f(p) = [(p \wedge p)^+] + O(p^3).$$

References

[1] S.K. Donaldson and P.B. Kronheimer. *The Geometry of Four-Manifolds*. New York: Oxford University Press, 1990. ISBN: 0198535538.

[2] H. Blaine Lawson Jr. and Marie-Louise Michelsohn. *Spin Geometry*. Princeton University Press, 1989. ISBN: 0691085420.