Lecture notes talk @ VU

j.w. Mélanie Theillière

h-principles near a submanifold

§1 Introduction

At its core, the field of h-principles is about proving that ~~a~~ certain maps between "spaces of geometric structures" are weak homotopy equivalences. This map is sometimes called a scanning map. Let's make this more concrete.

Geometric structures are often sections of some fibre bundle $X \to M$. The condition is then on the derivatives of that section. These derivatives are encoded coordinate-freely in the so-called rth-order jet bundle $J^r X \to M$. The condition then becomes a subset $R \subseteq J^r X$, called a partial differential relation (PDR)

For each section $f: M \to X$, we have an r-jet prolongation $j^r f: M \to J^r X$. The sections of this form of $J^r X \to M$ are called holonomic. For a PDR R, a ~~solution~~ solution is a holonomic section $j^r f: M \to R$. A formal solution is a section $F: M \to R$. We denote these spaces of sections by $\mathrm{Sol}(R)$ and $\mathrm{Sol}^f(R)$, respectively.

Examples of h-principles:
- Smale-Hirsch Immersion Theorem & Smale's sphere eversion
- Phillips' theorem for submersions.

In the rest of the talk, I will use Gromov's Op-notation: $\mathcal{O}p(A)$ means there exists some open neighborhood of A.

A tool for proving h-principles is holonomic approximation. The simplest version is:

<u>Theorem</u> (Eliashberg-Mishachev '01) Let $p: X \to M$ be a fibre bundle and let $A \subseteq M$ be a submanifold of positive codimension. Suppose that $F: \mathcal{O}p(A) \to J^r X$ is a section. Then there exists a C^0-small diffeotopy $h^t: M \to M$ and a holonomic section

$\hat{F} : \mathcal{O}p(h'(A)) \to J^r X$ such that $\hat{F}$ is C^0-close to F. + Picture!

For $r=1$, Massot and Theillière proved this (in 2021) using convex integration; this is another fundamental technique in proving h-principles. I wanted to give a similar proof for $r \geq 2$.

Philosophically, the difference between the two techniques is the source of the "flexibility": Holonomic approximation uses flexibility in the domain, whereas convex integration uses flexibility in the codomain.

The strategy is as follows: We split the problem into two parts:

- We study which sections of $(J^r X)|_A \to A$ extend holonomically (extension problem)
- We reduce the problem to one over A.

In the rest of this talk I will focus on the first step.

If we have a submanifold $A \subseteq M$, we have the following variation on h-principles:
We say that $\mathcal{R}$ satisfies an h-principle near A if for each section $F : \mathcal{O}p(A) \to \mathcal{R}$ there is a homotopy $H : \mathcal{O}p(A) \times [0,1] \to \mathcal{R}$ such that $H_0 = F$ and $H_1|_A$ is holonomic.

Such an h-principle only depends on data at A, if $\mathcal{R}$ is open and this we will make precise. We say that a section $F : A \to (J^r X)|_A$ admits a holonomic extension if there exists a holonomic section $\tilde{F} : \mathcal{O}p(A) \to J^r X$ extending F.

Morally, we have the following theorem:

<u>Theorem</u> Suppose $p : X \to M$ is a vector bundle and $A \subseteq M$ a submanifold of positive codimension. Then there exists a bundle $\overset{J^{r,\Delta}(A,X|_A)}{\cancel{\mathcal{Y}}} \to A$ with a subset of sections that are called holonomic and a bundle isomorphism $T : J^{\Delta,r}(A, X|_A) \to (J^r X)|_A$ such that holonomic sections of $J^{\Delta,r}(A, X|_A)$ correspond with holonomically extendable sections.

§2 Triangular jet bundle

We now shift focus to the following setting. Let $p: E \to A$ be a vector bundle (think tubular neighbourhood) and let $q: F \to A$ be a vector bundle. We can then look at ~~q^*~~ the pull-back bundle $p^*F \to E$ and then the restricted $(J^r p^*F)|_A$. The idea of the triangular jet bundle is to repackage the data in this restricted jet bundle.

Let us recall that a local model for the jet bundle is given by
$$J^r(\mathbb{R}^n, \mathbb{R}^q) \cong \mathbb{R}^n \times \mathbb{R}^q \times \mathrm{Pol}_0^{\leq r}(\mathbb{R}^n, \mathbb{R}^q) \quad \text{and} \quad \mathrm{Pol}_0^{\leq r}(\mathbb{R}^n, \mathbb{R}^q) \cong \bigoplus_{d=1}^{r} \underbrace{\mathrm{Pol}^d(\mathbb{R}^n, \mathbb{R}^q)}_{\text{homogeneous of degree } d}$$

Moreover, $\mathrm{Pol}^d(\mathbb{R}^n, \mathbb{R}^q) \cong \mathrm{Sym}^d((\mathbb{R}^n)^*) \otimes \mathbb{R}^q$.

Looking at the submanifold $A \subseteq E$ we get a "splitting" in directions: Those tangent to the submanifold and those tangent to it. On the level of vector spaces we could write $V = U \oplus W$. Then we have the following fact:
$$\mathrm{Sym}^d(V) = \mathrm{Sym}^d(U \oplus W) \cong \bigoplus_{k+l=d} \mathrm{Sym}^k(U) \otimes \mathrm{Sym}^l(W).$$
We induce a similar splitting on the level of jet bundles.

<u>Definition</u> Let $r \geq 1$. The Taylor bundle on E of degree d with values in F is
$$P_r(E,F) = \bigoplus_{d=0}^{r} (\mathrm{Sym}^d(E^*) \oplus F).$$

The name comes from the following:

<u>Definition</u> Let $f: E \to F$ be a C^∞ map over A. Then the fibre derivative at $x \in E$ in direction $v \in E_{p(x)}$ is
$$Df(x)v := \frac{d}{dt}\Big|_{t=0} f(x+tv).$$

This gives again a map $Df: E \to E^* \otimes F$ over A and so one can take higher derivatives $D^r f: E \to \mathrm{Sym}^r(E^*) \otimes F$.

<u>Lemma</u> Evaluation on vectors gives an inclusion of $C^\infty(A)$-module sheaves
$$\Gamma(P_r(E,F)) \hookrightarrow \Gamma(p_* F \to E), \quad \sigma \mapsto \hat{\sigma}$$
~~pullback~~ pushforward sheaf along p.

Then it follows that

Lemma The inclusion from the previous lemma induces a surjective vector bundle morphism

$$P: J^r P_r(E,F) \longrightarrow J^r(p^*E)|_A, \quad j^r_x\sigma \longmapsto j^r_{0_x}\hat{\sigma}.$$

The proof is a computation in local coordinates. The key observation during the proof is that P is essentially a projection map where the value of the degree-d part along the fibre only depends on the $r-d$ jet of the section. We therefore define

Definition Let $r \geq 1$ be an integer. Then the triangular jet bundle of order r on E with values in F is

$$J^{\Delta,r}(E,F) = \bigoplus_{k+d=r} J^k(\mathrm{Sym}^d(E^*)\otimes F).$$

Then we get

Theorem The surjective vector bundle morphism P descends to a vector bundle isomorphism

$$T: J^{\Delta,r}(E,F) \longrightarrow J^r(p^*F)|_A.$$

The proof consists of showing that P indeed descends along some submersion and a combinatorial argument.

Since the triangular jet bundle is a fibre product of $r+1$ jet bundles, it has a natural notion of a holonomic section:

Definition Let $\pi_d: J^{\Delta,r}(E,F) \to J^{r-d}(\mathrm{Sym}^d(E^*)\otimes F)$ denote the projection map $(0 \leq d \leq r)$. We say that a section F is holonomic if $\pi_d \circ F$ is holonomic for $0 \leq d \leq r$.

Note that a section of J^0 is always holonomic, so we could have asked for $0 \leq d < r$. The following result explains the significance of the triangular jet bundle:

Theorem Let $F: A \to (J^r p^*F)|_A$ be a section. Then F admits a holonomic extension if and only if $T^{-1} \circ F$ is holonomic.

§3 Correspondence between h-principles

Let us finally connect this back to h-principles near submanifolds. Let $X \to M$ a vector bundle, $A \subseteq M$ a submanifold of positive codimension, and $R \subseteq J^r X$ a PDR. We can pick a ~~try~~ tubular ~~nt~~ neighbourhood $(U, E \to A, \Psi)$ where $A \subseteq U$ open, $E \to A$ a vector bundle and $\Psi : E \hookrightarrow \cancel{A} M$ an embedding ~~such~~ that $\Psi(E) = U$. Let $F := X|_A$, and let R_U be the restriction of R to U. Then we have a chain of isomorphisms

$$(J^r X)|_{\cancel{A}U} \cong J^r(X|_U) \cong J^r(p^* F) \rightsquigarrow (J^r X)|_A \cong J^r(p^* F)|_A \cong J^{\Delta, r}(E, F).$$

and so R corresponds to some $\widetilde{R} \subseteq J^{\Delta, r}(E, F)$.

Then we have:

<u>Theorem</u> Let $R \subseteq J^r X$ be open. Then R satisfies an h-principle near A if and only if $\widetilde{R} \subseteq J^{\Delta, r}(E, F)$ satisfies an h-principle.